I. Introduction and Poset Definitions

A. Introduction

In this project, we were primarily focused on testing a conjecture pertaining to posets. The conjecture concerns the relationship between posets with the jdt property and posets with the Littlewood-Richardson property. In order to perform such tests, we used *Mathematica 5.1*. While identifying and comparing posets with these two properties was our main goal, an equally important aspect of the project is the collection of poset list files created by our programs. If a user is familiar with the data structures we use to represent posets in Mathematica, then these files provide this user with reliable lists of posets with various properties. These lists are easily accessible and have been tested for accuracy.

A further important by-product of our project is a system of obtaining a unique name for each poset. Mathematica first provided a means to identify a standard form for each isomorphism class of posets. With this and Mathematica's sort feature, we were able to find a standard ordering of the standard forms for each poset size. Referencing the poset by its size and the position of its standard form in this list gives the desired unique name. For example, since the standard form of the 4-element chain is the 16^{th} poset in the standard ordering for size 4, we can refer to this poset as 4.16.

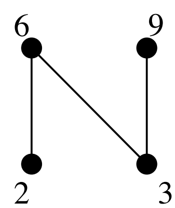
In this paper, we explain all that is needed to understand the results of our project. We begin with the theory needed to examine posets, in particular posets with the two previously mentioned properties. Next we explain how we translated mathematical theory into the Mathematica environment. So that the reader may better understand the programs and files created, the organization of the programming project is then described. This is followed by descriptions of all our programs and data files. Finally we present the results we obtained.

B. Basic Poset Definitions

Throughout this paper we will be concerned with different properties that certain structures called posets may possess. Before we introduce these properties, we begin with some basic definitions.

A partially ordered set or **poset** is a set P together with a partial ordering, ≤. That is, ≤ is a reflexive, antisymmetric and transitive relation. The **size** of this poset is the cardinality of the set P. For the remainder of the paper, P is assumed to be a finite poset. We use the convention that x < y means x ≤ y and x ≠ y. For two elements x, y in P, y is said to **cover** x if x < y and if there is no z in P such that x < z < y. If neither x ≤ y nor y ≤ x, x and y are said to be **incomparable**. An example of a poset of size n is the integers 1 through n together with its usual order. The notation **n** is used to represent this poset.

Posets are depicted with **Hasse diagrams**. The Hasse diagram of a poset shows the poset's covering relations. The Hasse diagram of P is formed by representing each element of P with a dot. These dots are arranged such that if y covers x, x is drawn below y and the two dots are connected with a line. For example, consider the set P={2,3,6,9}. Division gives a partial ordering for this set: x ≤ y if x divides y. With the dots labeled for clarity, the Hasse diagram for this poset is:

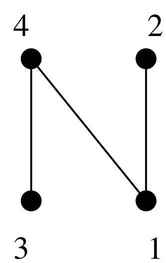


A **connected poset** is a poset whose Hasse diagram is a connected graph. As in graph theory, if y covers x in P, we call x a **child** of y and refer to y as a **parent** of x. Often we consider Hasse diagrams whose dots are labeled with integers. A **labeled poset on [n]** is a partial ordering on n: The elements in the Hasse diagram are labeled with the integers 1 to n. The poset is **naturally labeled** if i < j in P implies that i < j as integers. Visually, the label of a dot must be larger than the labels of its children. In Section II.B, we specify a standard natural labeling for each poset.

Within a poset, there are maximal and minimal elements. A **maximal element** in P is an element that is not covered by any other elements. Similarly, a **minimal element** is one that does not cover any other elements. Hasse diagrams make maximal and minimal elements particularly easy to identify. Minimal elements have no lines below them and maximal elements have no lines drawn above them in the Hasse diagram. In the above example, 2 and 3 are minimal elements while 6 and 9 are maximal. At times we will consider posets with a unique maximal element. Observe that if a poset is not connected, each connected component of its Hasse diagram will have at least one maximal element. So posets with a unique maximal element are necessarily connected.

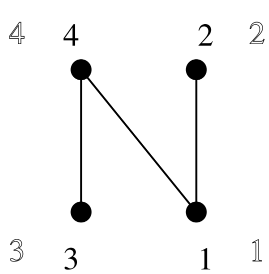
There are also certain subsets of posets that we will examine. A **chain** is a subset C of P such that for every x and y in C, either $x \le y$ or $y \le x$. An **antichain** is a subset A of P such that any two distinct elements of A are incomparable. An **ideal** is a subset I of P such that if x is in I and $y \le x$ then y is also in I. A subset S of P is said to generate I if I is the smallest ideal in P that contains S. For a poset P, there is a one-to-one correspondence between antichains and ideals. If I is an ideal of P, then the set A of the maximal elements of I is an antichain. Forming the ideal generated by this set A gives back the original ideal I. A **filter** is a subset F of P such that if x is in F and $y \ge x$ then y is also in F. Note that if I is an ideal then the elements in P-I form a filter. For any x,y in P, the **interval** [x,y] is the set of all z in P such that $x \le z \le y$. Convex sets are another type of subset that we will consider. A subset C of P is **convex** if for x and y in C, if z is any element such that $x \le z \le y$, then z is also in C.

In our examination of posets, we also consider functions from one poset to another. A function φ from the poset P to the poset Q is **order-preserving** if $x \le y$ in P implies that $\varphi(x) \le \varphi(y)$ in Q. Suppose the size of P is n. Then an **order extension** of P is an order-preserving bijection, φ, from P to the poset **n**. Suppose P is labeled. Let's view its labels $b_1, b_2, \ldots, b_n$ as being black. And let's regard elements of the codomain **n** as being yellow. We can represent an order extension of P using a one-rowed permutation by forming $(\varphi(b_1), \varphi(b_2), \ldots, \varphi(b_n))$, which is a list of yellow labels. So φ gives a new yellow labeling of P. This yellow labeling is found by relabeling the element b_i with $\varphi(b_i)$ in the Hasse diagram of P for $1 \le i \le n$. For example, consider the following naturally labeled poset P:

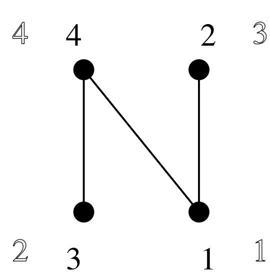


Then the order extensions and corresponding yellow-labeled Hasse diagrams of P are:

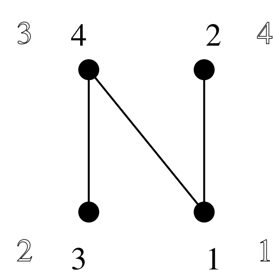
(1,2,3,4)



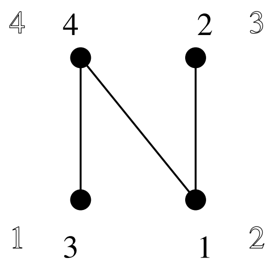
(1,3,2,4)



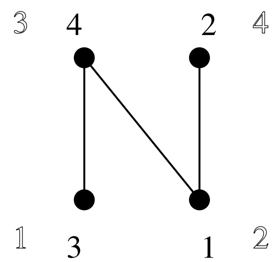
(1,4,2,3)



(2,3,1,4)



(2,4,1,3)



More often, we consider the inverse of one of these order extensions, an **inverse extension**. Again view P as being labeled in black and view the elements of **n** as being yellow. As with order extensions, we can represent an inverse extension of P using a one-rowed permutation. To do this, we form the list ($\varphi^{-1}(1)$, $\varphi^{-1}(2)$, …, $\varphi^{-1}(n)$). This is a list of black labels of P, which is obtained by reading the black labels according to the yellow order. For example, if P is the poset above, the inverse extensions of P are the lists of black numbers (1,2,3,4), (1,3,2,4), (1,3,4,2), (3,1,2,4), and (3,1,4,2).

Two posets P and Q are **isomorphic** if there is a bijection, φ, from P to Q such that $x \le y$ in P if and only if $\varphi(x) \le \varphi(y)$ in Q; that is, if both φ and φ^{-1} are order preserving. Observe that all posets within an isomorphism class have the same unlabeled Hasse diagram, and distinct classes have distinct unlabeled Hasse diagrams. Hence unlabeled Hasse diagrams correspond exactly with isomorphism classes of posets, which are often called unlabeled posets or simply posets for brevity. Then if we have a labeled poset on [n], an inverse extension gives a rule for relabeling the poset to obtain an isomorphic naturally labeled poset. In fact, if we find all the inverse extensions of a naturally labeled poset, we can obtain all naturally labeled posets in its isomorphism class. Often, the relabeling is injective; that is, each inverse extension gives rise to a different poset. However, this is not always the case. For example, consider the poset of size four consisting of a single antichain. Any inverse extension of this poset will give the same naturally labeled poset.