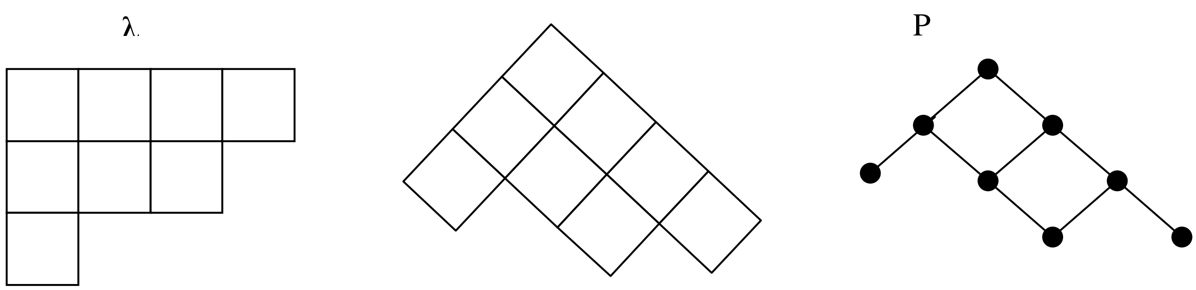
II. Poset Properties

In this chapter we will give explanations of three properties that posets may possess. We describe the d-complete property in section D. The other two properties are known as the jdt property and the Littlewood-Richardson property. Both of these properties originated in the context of skew Young tableaux. Jeu de taquin is a process that produces a standard Young tableau from a skew standard Young tableau by repeatedly shifting its entries. A standard Young diagram has an underlying poset. For example, consider the following Young diagram λ. To find its corresponding poset P, we rotate λ by 45° and replace each box with a point. Two such points are connected with a line if they represent adjacent boxes.



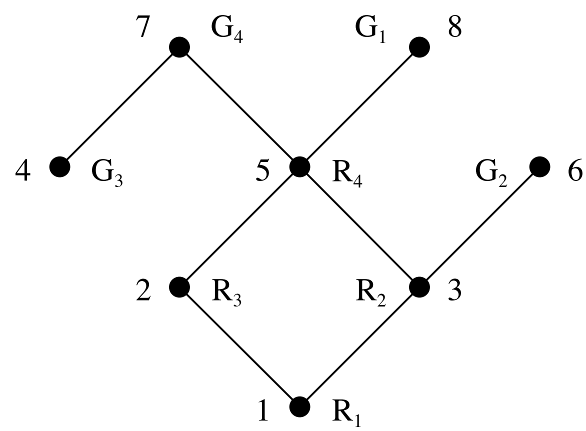
As a result of the correspondence between Young diagrams and posets, aspects of the jeu de taquin (jdt) process can be extended to posets. Schützenberger proved that Young diagrams have the jdt property [Fu].

Classically, the Littlewood-Richardson rule concerns the multiplicities of Schur functions, which can be described with semi-standard Young tableaux. Let $n \geq 1$ and let λ be a Young diagram of size n. The Schur function of λ, $s_\lambda(x_1,\ldots,x_n)$, is a symmetric polynomial in $x_1,\ldots, x_n$. The Littlewood-Richardson rule is used to calculate the product of two Schur functions, $s_\lambda(x_1,\ldots,x_n)$ and $s_{\square}(x_1,\ldots,x_n)$, each indexed by a Young diagram. One proof of the Littlewood-Richardson rule uses the jdt property for Young diagrams ([BSS], example 3.3). Proctor produced the definition of the Littlewood-Richardson property for posets as a way of distilling out the most essential part of the proof of the Littlewood-Richardson rule for Schur functions. Before we consider these concepts in the realm of posets, we first give preliminary definitions and operations.

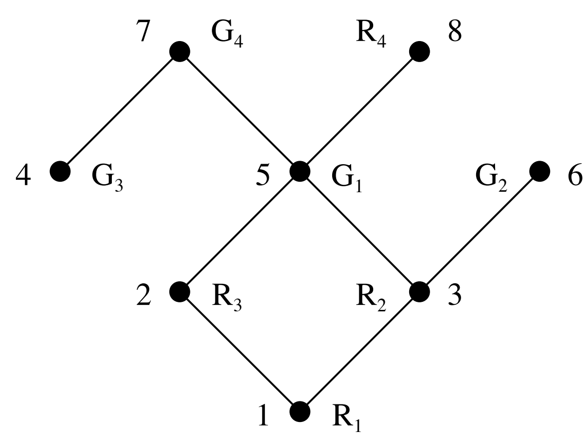
A. Move Operations

Let P be a fixed poset. In the following discussion we will consider two totally ordered sets: $R = \{R_1 < R_2 < R_3 < \ldots\}$ and $G = \{G_1 < G_2 < G_3 < \ldots\}$. We will refer to the elements of R as **red labels** and the elements of G as **green bubbles**.

Now let $N \subseteq P$ and $k = |N|$. We refer to an order preserving bijection $Y: N \to \{R_1, R_2, \ldots, R_k\}$ as a **red numbering** on P. Similarly, if $T \subseteq P$ and $|T| = m$, we refer to an order preserving bijection $B: T \to \{G_1, G_2, \ldots, G_m\}$ as a **green numbering** on P. We refer to the set N as the **shape** of Y and the set T as the shape of B. The pair (B,Y) is a **bi-numbering** of P if $T \cup N = P$ and $T \cap N = \varnothing$. For example the following is a black labeled 8-element poset P with a bi-numbering, (B,Y):



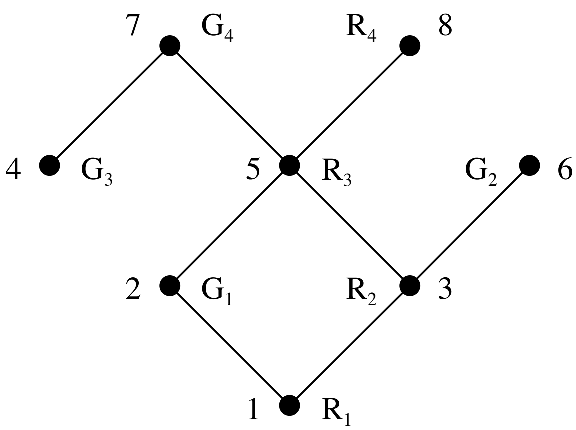
Given a bi-numbering (B,Y) of P, we can define **move operators**. The move operator $\boldsymbol{M_1}$(B,Y) gives a new green numbering, A, and a new red numbering, X, on P. The numbering A is produced by moving the green bubble G_1 to the location of the largest red label that it covers. Since a red label is moved in this process, M_1 also produces the red numbering X on P. So the result of M_1(B,Y) is a new bi-numbering (A,X) on P. If we consider the initial bi-numbering of the poset above, M_1 produces the bi-numbering, (A,X):



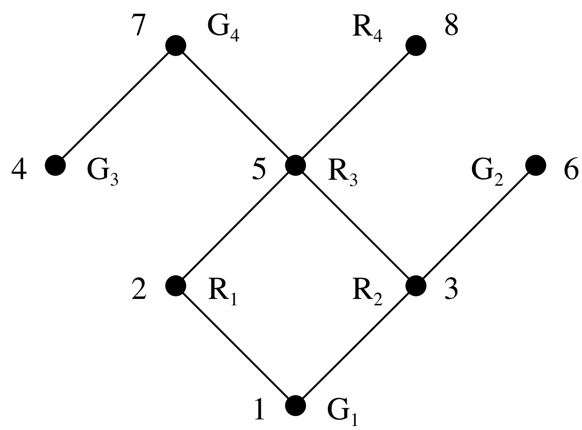
We can similarly define a move operator $\boldsymbol{M_k}$ by moving the green bubble G_k.

We obtain another operator from each of these move operators. The **pull operator**, $\boldsymbol{Pull_k}$, is defined by iterating M_k until G_k does not cover any red labels. For example, consider again the black labeled poset P we started with above. After one application of M_1, we obtained the bi-numbering (A, X). Applying M_1 a couple more times gives the following progression of bi-numberings:

(C,Z)



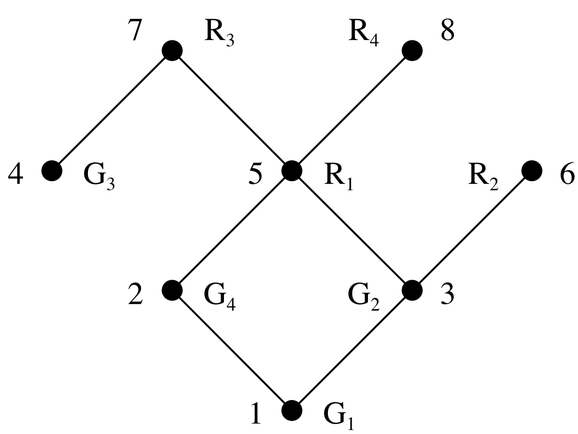
(D,W)



Therefore the result of $Pull_1$(B,Y) is the bi-numbering (D,W). We can think of the operation $Pull_k$ as 'sliding' the green bubble G_k down as far as possible. For a given bi-numbering (B,Y) on P, we can define ***Pull***(B,Y) to be the bi-numbering obtained by successively applying $Pull_1, Pull_2, \ldots, Pull_m$. That is,

$$Pull(B,Y) = Pull_m \circ \ldots \circ Pull_2 \circ Pull_1(B,Y).$$

Again consider the poset P we started with above. Then *Pull*(B,Y) is

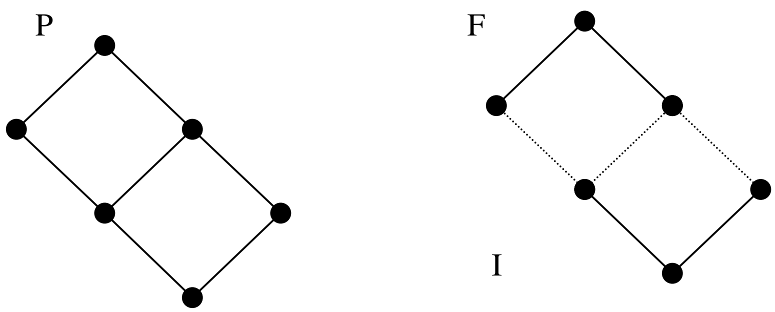


For the forthcoming properties, we will need to separately consider the resulting red numberings and the resulting green numberings of *Pull*(B,Y). We will use $\boldsymbol{Pull}_r$(B,Y) to denote the upper red numbering that arises from the pull operation. Similarly, we will refer to the lower green numbering that is produced as $\boldsymbol{Pull}_g$(B,Y).

B. The jdt Property

With these operations, we can now define the jdt property for posets. The jdt property is defined in Proctor's paper [Pr1]. The following is taken from a restatement [Pr2] of that definition. A poset P has the **jdt property** if and only if for <u>every</u> filter F of P and for <u>every</u> red numbering Y of the complementary ideal P–F and for <u>every</u> green numbering B of F, we have that $Pull_r$(B,Y) = $Pull_r$(C,Y) for <u>every</u> green numbering C of F.

In determining whether or not a poset has the jdt property, we consider every possible filter F and corresponding ideal I of P. To illustrate this property, we consider the following poset P and ideal/filter pair I/F.

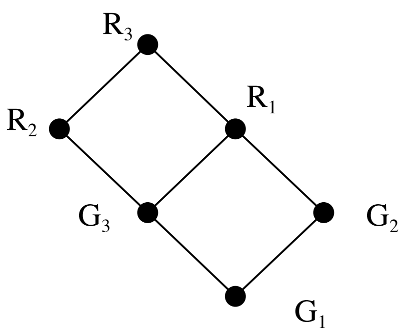


For each ideal/filter pair, all red numberings of I and all green numberings of F are found. For example, the red numberings of the ideal I shown above are:

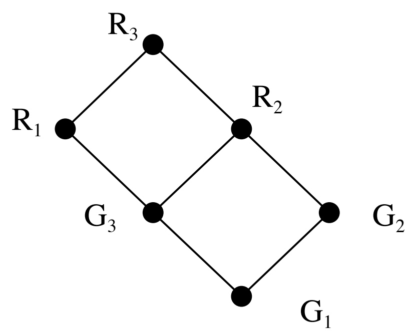
And the green numberings of the filter F shown above are:

For every such red numbering Y and green numbering B, the operation *Pull*(B,Y) is performed. For our continuing example, the operation *Pull* is performed four times, giving the following resulting bi-numberings:

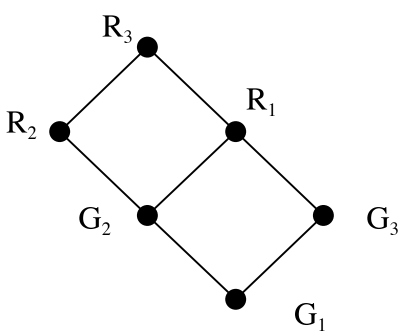
$(A,Z) = Pull(B_1,Y_1)$



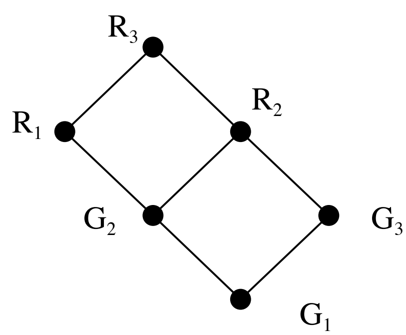
$(C,X) = Pull(B_1,Y_2)$



$(D,W) = Pull(B_2,Y_1)$



$(E,V) = Pull(B_2,Y_2)$



For a given red numbering Y, the resulting red numberings, $Pull_r(B,Y)$, are compared for all B. If they are all the same for every Y, the poset has the jdt property. If there is some Y such that the red numbering $Pull_r(B,Y)$ and the red numbering $Pull_r(C,Y)$ are different for some green numberings B and C of F, then the poset does not have the jdt property. For our example, the resulting red numberings in the first column, Z and W, are the same and the resulting red numberings, X and V, in the second column are the same. Therefore the conditions for the jdt property are satisfied for this ideal and filter of P. To determine if P has the jdt property, all other ideal/filter pairs must next be considered. We refer to a poset that has the jdt property as a **jdt poset**.

C. The Littlewood-Richardson Property

Before we present the next poset property, we will introduce some notation for sets related to numberings and the shove operator. Let F be a filter of P and I = P – F. Then ***S*(F)** will denote the set of all red numberings of F and ***S*(I)** will denote the set of all red numberings of its corresponding ideal. For example, consider again the poset P and ideal/filter pair, I/F, from the previous section. Then $S(\mathrm{I}) = \{\mathrm{Y}_1, \mathrm{Y}_2\}$. For each fixed green numbering B of F, we can define the set of resulting bi-numberings

$\boldsymbol{O}_{\mathbf{B}}(\mathbf{I})$ = {(E,U): (E,U) = *Pull*(B,Y) for some Y in *S*(I)}.

Let B be a green numbering of F. To form $O_{\mathrm{B}}(\mathrm{I})$ we consider all red numberings, Y, of the ideal I. For each of these, the operation *Pull*(B,Y) is performed. The resulting bi-numberings compose $O_{\mathrm{B}}(\mathrm{I})$. For our continued example, $O_{\mathrm{B1}}(\mathrm{I})$ = {(A,Z), (C,X)} and $O_{\mathrm{B2}}(\mathrm{I})$ = {(D,W), (E,V)}. We will now group numberings together according to the filter shapes N of the resulting red numberings:

$\boldsymbol{O}_{\mathbf{B}}(\mathbf{I,N})$ = {(E,U) : (E,U) = *Pull*(B,Y) for some Y in *S*(I) and the shape of U is N},

$\boldsymbol{GO}_{\mathbf{B}}(\mathbf{I,N})$ = {E : E = $Pull_{\mathrm{g}}$(B,Y) for some Y in *S*(I) and the shape of E is P–N}.

For our example, for all green numberings B of F, the only filter N for which $O_{\mathrm{B}}(\mathrm{I,N}) \neq \varnothing$ is N = F. For that example, $O_{\mathrm{B1}}(\mathrm{I,F})$ = {(A,Z), (C,X)} and $O_{\mathrm{B2}}(\mathrm{I,F})$ = {(D,W), (E,V)}. Also, $GO_{\mathrm{B1}}(\mathrm{I,F})$ = {A,C} = {A} since C = A and $GO_{\mathrm{B2}}(\mathrm{I,F})$ = {D,E} = {D} since E = D.

We are now ready to define the Littlewood-Richardson Property. This definition is an improved version of the one given in Proctor's Ann Arbor talk [Pr2]. However it was shown [Pr3] to be equivalent. A poset P has the **Littlewood-Richardson property** if and only if, for every filter F of P, there is at least one fixed green numbering B of F, such that

$O_{\mathrm{B}}(\,\mathrm{P–F}\,) = \cup_{\mathrm{N}}\, GO_{\mathrm{B}}(\,\mathrm{P–F, N}\,) \times S(\mathrm{N})$,

where the union is over all filters $\mathrm{N} \subseteq \mathrm{P}$ with $GO_{\mathrm{B}}(\,\mathrm{P–F, N}) \neq \varnothing$

As with the jdt property, to determine if a poset P has the Littlewood-Richardson property we consider every filter F of P and its corresponding ideal P–F. Then we find each green numbering B of F. For every red numbering Y of the ideal P–F, we perform the operation *Pull*(B,Y). For our example, this gives us the 2 × 2 array of bi-numberings shown at the end of the previous section. The shape of each of the resulting green numberings, $Pull_{\mathrm{g}}$(B,Y) is an ideal, G, of P. For each of these, we find all red numberings of the corresponding filter N = P–G to form the set *S*(N). For our example the only such shape G that arises is G = I. Then N = F and we have *S*(F) = {Z, X}. To test a poset for the Littlewood-Richardson rule, we then check to see if $\cup_{\mathrm{N}}\, GO_{\mathrm{B}}(\,\mathrm{P–F, N}\,) \times S(\mathrm{N})$ is equal to $O_{\mathrm{B}}(\mathrm{I})$. Returning again to our example, we have

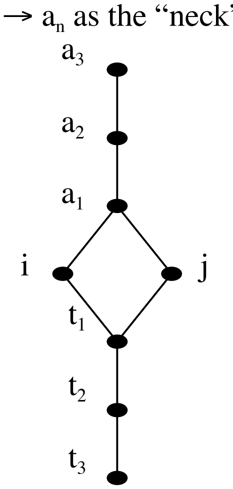
$\cup_{\mathrm{N}}\, GO_{\mathrm{B1}}(\,\mathrm{P–F, N}\,) \times S(\mathrm{N}) = GO_{\mathrm{B1}}(\,\mathrm{P–F, F}\,) \times S(\mathrm{F})$ = {(A,Z), (A,X)} = $O_{\mathrm{B1}}(\mathrm{I})$, since C = A.

Therefore B_1 satisfies the condition for the Littlewood-Richardson conjecture for this ideal/filter pair. The second bi-numbering B_2 also satisfies the condition. If these are equal for at least one green numbering B for every filter F of P, then we say that P is a **L-R poset**.

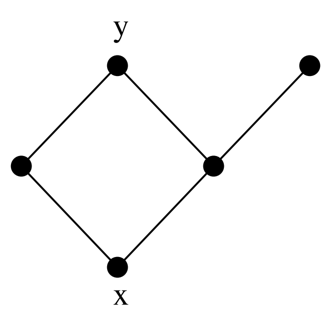
D. d-complete Posets

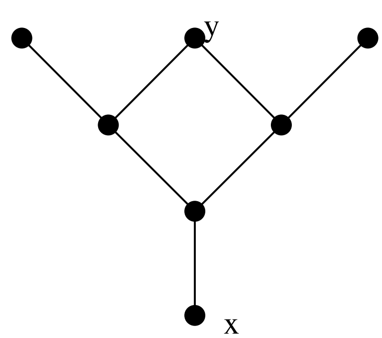
In 1993, Proctor developed the d-complete property [Pr4] while taking a combinatorial approach to certain theorems in Representation theory. Posets that have the d-complete property satisfy certain local structural conditions. In order to understand these conditions, we must begin with some related definitions. These definitions as well as the definition of d-complete come from the paper [Pr1] on the jdt property. However Proctor's recent improvement [Pr3] of the d-complete definition is incorporated.

Let b,n ≥ 0. The poset $\square_{b,n}$ consists of two incomparable elements i and j together with a chain of b elements $t_b \to \dots \to t_2 \to t_1$ such that $t_1 \to i$ and $t_1 \to j$ and a chain of n elements $a_1 \to a_2 \to \dots \to a_n$ such that $i \to a_1$ and $j \to a_1$. We refer to the chain $t_b \to \dots \to t_2 \to t_1$ as the "tail" and the chain $a_1 \to a_2 \to \dots \to a_n$ as the "neck". For example, the following poset is $\square_{3,3}$:

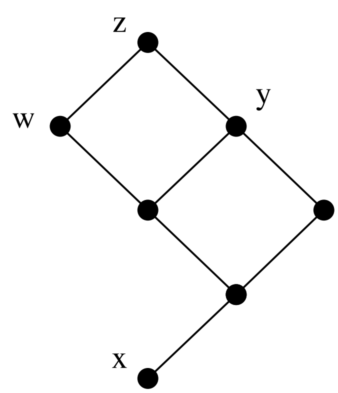


Now we turn our attention from the $\square_{b,n}$ posets to related subsets of posets. Let P be a poset and k ≥ 3. An interval [x,y] in P is a **d_k-interval** if it is isomorphic to $\square_{k-2,k-2}$. A subset of P is a **d^-_k-subset** if it is isomorphic to $\square_{k-2,k-3}$. For example the poset on the left below has a d_3-interval and in the poset on the right, the interval [x,y] is a d^-_4-subset.

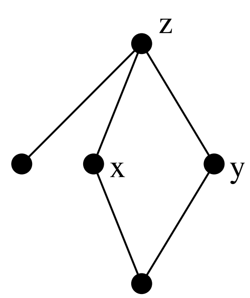




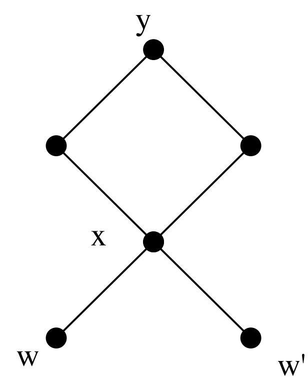
Because of their appearance, we call d_3-intervals "diamonds". Observe that for all $k \geq 3$, every d_k-interval contains a diamond. A d^-_3-subset consists of two incomparable elements x and y which both cover an element w. Then a d^-_3-subset in P is **completed** if there exists an element z in P such that z covers x and y and [x,z] is a d_3-interval. For $k > 3$, a d^-_k-subset (and interval) [x,y] is said to be **completed** if there exists a z in P such that z covers y and [x,z] is a d_k-interval. In the poset shown below, [x,y] is a d^-_4-subset that is not completed since [x,z] is not isomorphic to $\square_{2,2}$.



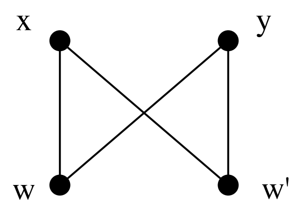
Let S be a d^-_k-subset. We say S is completed **freely** if any z completing S does not cover any elements outside of S. The d^-_3-subset in the poset below is not freely completed.



Now let [x,y] be a d_{k-1}-interval in P for $k > 3$. Suppose that x covers two elements w and w' and that [w,y] and [w',y] are both d^-_k-subsets. Then [w,y] and [w',y] are called **overlapping d^-_k-subsets**. The figure below has overlapping d^-_4-subsets.

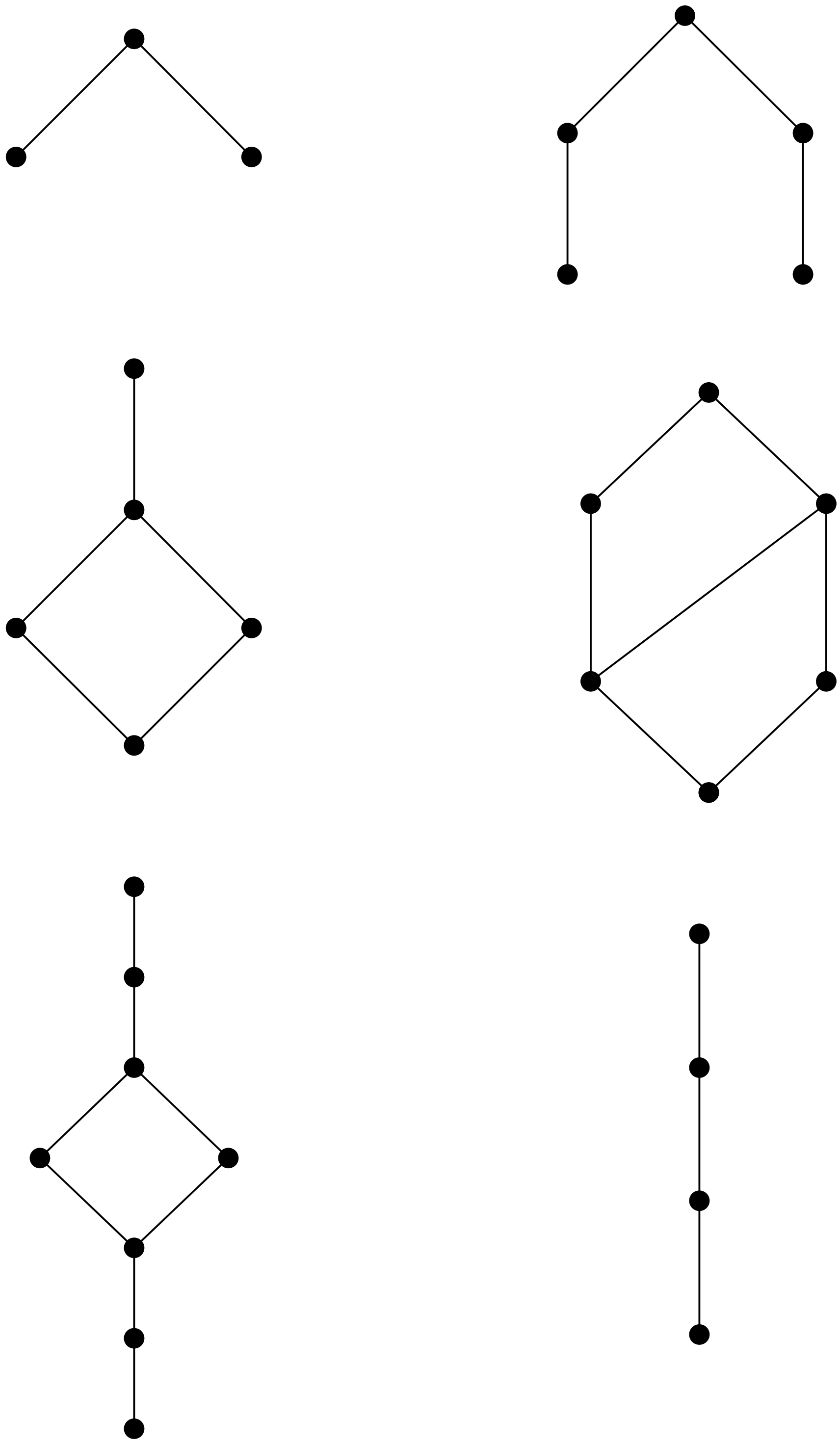


We can also define overlapping d^{-}_{3}-subsets. Suppose we have two d^{-}_{3}-subsets: one consisting of the incomparable elements x and y that cover w and one consisting of incomparable elements x' and y' that cover w'. These two d^{-}_{3}-subsets **overlap** if x = x', y = y', and w ≠ w'. The poset below has overlapping d^{-}_{3}-intervals. This is the complete bipartite graph on two sets of two vertices, $K_{2,2}$.



We are now ready to define the d-complete property. A poset P is **d-complete** if for every k ≥ 3 each d^{-}_{k}-subset is completed freely and there are no overlapping d^{-}_{k}-subsets. Since a d-complete poset has only completed d^{-}_{3}-subsets, any v-shaped subset of the poset is the bottom of a diamond. Furthermore every diamond in the poset that has a tail also has a neck of the same length because all d^{-}_{k}-subsets in the poset are completed for k > 3. Since every d^{-}_{k}-subset in a d-complete poset is completed freely, elements that are the tops of diamonds only cover two elements. Also this condition ensures that any other element in a d-complete poset that is in the neck of a d_k-interval can only cover one element. Finally, overlapping d^{-}_{3}-subsets create a $K_{2,2}$ in a poset. Therefore d-complete posets have no $K_{2,2}$-shapes.

Here are some examples of d-complete posets:



E. Proctor's Conjecture

Now that we have defined them, it is natural to speculate about the relationship between posets with these three properties. As mentioned in the introduction, a main component of this project is comparing jdt posets with L-R posets. The following result previously established [Pr2] a relationship between these families of posets.

Theorem 2.1

Let P be a poset. If P has the jdt property, then P has the Littlewood-Richardson property.

After obtaining this theorem in 2003, Proctor asked if the converse is also true. Because of the computational results described in Chapter VII, Proctor has now made the following conjecture.

Conjecture

Let P be a poset. If P has the Littlewood-Richardson property, then P has the jdt property.

In this project we have confirmed this conjecture for posets with no more than 9 elements. As the sizes increase, the number of posets to examine becomes very large. For this reason, we use one of Proctor's earlier results from studying posets with the jdt property [Pr1].

Theorem 2.2

If P is a d-complete poset then it has the jdt property.

Observe that as an immediate consequence of Theorem 2.2 and Theorem 2.1, a d-complete poset also has the Littlewood-Richardson property. So all d-complete posets can be removed before testing a list of posets to see which have the jdt property and which have the Littlewood-Richardson property. Therefore the process of confirming the conjecture for a given poset size can be shortened. In the following chapters, we will describe our testing process for this conjecture.

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